

# On maps which preserve equality of distance in $F^*$ -spaces<sup>1</sup>

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**Abstract** In order to generalize the results of Mazur-Ulam and Vogt, we shall prove that any map  $T$  which preserves equality of distance with  $T(0) = 0$  between two  $F^*$ -spaces without surjective condition is linear. Then , as a special case linear isometries are characterized through a simple property of their range.

## 1 Preliminaries and introduction

Recall from [1] that a non-negative function  $\|\cdot\|$  defined on a linear space  $E$  is called an  $F$ -norm provided

- (i)  $\|x\| = 0$  iff  $x = 0$ ;
- (ii)  $\|ax\| = \|x\|$  for all  $a$  with  $|a| = 1$ ;
- (iii)  $\|x + y\| \leq \|x\| + \|y\|$ ;
- (iv)  $\|a_n x\| \rightarrow 0$  provided  $a_n \rightarrow 0$ ;
- (v)  $\|ax_n\| \rightarrow 0$  provided  $x_n \rightarrow 0$ .

An  $F$ -norm  $\|\cdot\|$  induces a transitive invariant distance  $d$  by

$$d(x, y) = \|x - y\| \quad \forall x, y \in E.$$

A linear space  $E$  with an  $F$ -norm  $\|\cdot\|$  is said to be an  $F^*$ -space. The complete  $F^*$ -space is called  $F$ -space.

Let  $E = (E, \|\cdot\|)$ ,  $F = (F, \|\cdot\|)$  be two  $F^*$ -spaces, Let  $R_0^+$  denote the set

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<sup>1</sup>**Keywords**  $F^*$ -space, equality of distance preserving map, isometry

**2000 AMS Subject Classification** 46A16.

This paper is supported by The National Natural Science Foundation of China (10571090) and The Research Fund for the Doctoral Program of Higher Education (20010055013).

of non-negative real numbers. We say that a map  $T : E \rightarrow F$  preserves equality of distance iff there exists a function  $\phi : R_0^+ \rightarrow R_0^+$  such that

$$\|Tx - Ty\| = \phi\|x - y\|, \quad \forall x, y \in E.$$

The function  $\phi$  is called the gauge function for  $T$ , and this definition equivalently means that

$$u, v, z, w \in E, \quad \|u - v\| = \|z - w\| \Rightarrow \|Tu - Tv\| = \|Tz - Tw\|.$$

In particular, when  $\phi$  is the identity function, mapping  $T$  is an isometry. Such mappings were studied by Schoenberg [2] and by Von Neumann and Schoenberg [3] for Hilbert spaces.

The classical theorem of Mazur-Ulam [4] states that an onto isometry between two real normed spaces is affine, Charzyński [5] and Rolewicz [6] have shown, respectively, that surjective isometries of finite-dimensional  $F$ -spaces and of locally bounded spaces with concave norm are also linear. Ding and huang [8] showed that Rolewicz's result also holds in locally midpoint constricted  $F$ -spaces. More generally, Vogt [7] has shown that equality of distance preserving maps between two real normed spaces are linear. The hypothesis of surjectivity of the maps was required in the results mentioned above.

The present paper will extend the result of Vogt to a large class of  $F^*$ -spaces including all  $p$ -normed spaces ( $0 < p \leq 1$ ) without surjective condition, but needing a simple peculiar property of the maps' range. The proof of the main result here depends on the technique of Vogt and all the spaces mentioned in this paper are assumed to be real.

## 2 Main results

The following lemma is in metric space theory, which was adapted by Vogt in [7] and similar to one stated by Aronszajn[9].

**Lemma 2.1.** Let  $(M, d)$  be a bounded metric space. Suppose there exists an element  $m$  in  $M$ , a surjective isometry  $V : M \rightarrow M$ , a constant  $K > 1$ , such that for all  $x$  in  $M$ ,  $d(Vx, x) \geq Kd(m, x)$ . Then  $m$  is a fixed point for every surjective isometry  $S : M \rightarrow M$ .

**Proof.** Since metric space isometries are injective,  $V^{-1}$  and  $S^{-1}$  exist, and  $V, S, V^{-1}$  and  $S^{-1}$  are bijective isometries of  $M$  together with arbitrary (finite) compositions of them.

Define a sequence of isometries  $V_n : M \rightarrow M$  and elements  $m_n$  in  $M$  indexed by the integers  $n \geq 1$ ,

Let

$$\begin{aligned} V_1 &= V, & m_1 &= m, \\ V_2 &= SVS^{-1}, & m_2 &= Sm, \\ V_{n+1} &= V_{n-1}V_n(V_{n-1})^{-1}, & m_{n+1} &= V_{n-1}m_n, \quad n \geq 2. \end{aligned}$$

Each  $V_n$  is a bijective, invertible isometry of  $M$  and a straight-forward induction yields:

$$d(V_n x, x) \geq Kd(m_n, x) \quad x \in M, n \geq 1. \quad (1)$$

If we let  $x = m_{n+1}$  in (1), we obtain

$$d(m_{n+2}, m_{n+1}) = d(V_n m_{n+1}, m_{n+1}) \geq Kd(m_n, m_{n+1}) = Kd(m_{n+1}, m_n).$$

Another induction gives

$$d(m_{n+2}, m_{n+1}) \geq K^n d(m_2, m_1) \quad \forall n \geq 1.$$

Since  $M$  is a bounded metric space, there exists a positive number  $N$  such that

$$d(m_{n+2}, m_{n+1}) \leq N \quad \forall n \geq 1.$$

Hence

$$d(m_2, m_1) \leq N/K^n \quad \forall n \geq 1.$$

Since  $K > 1$ , we have  $d(m_2, m_1) = 0$ . This implies

$$Sm = m_2 = m_1 = m. \quad \square$$

Rassias [12] shows that the ratio  $\|2x\|/\|x\|$  plays an important role in the generalizations of the Mazur-Ulam theorem and we shall show that the ratio  $\|2x\|/\|x\|$  is also of great importance in the following statement.

**Theorem 2.2.** Let  $E, F$  be  $F^*$ -spaces, and there exists a positive number  $r$  such that  $\alpha_F(r) = \inf\{\frac{\|2x\|}{\|x\|} : x \in F, 0 < \|2x\| \leq r\} > 1$ , Let  $T : E \rightarrow F$  with  $T(0) = 0$  be a continuous map which preserves equality of distance, i.e. there exists  $\phi : R_0^+ \rightarrow R_0^+$  such that

$$\|Tx - Ty\| = \phi\|x - y\|, \quad \forall x, y \in E.$$

and satisfy

$$2Tx - Ty \in T(E), \quad \forall x, y \in E.$$

Then  $T$  is linear.

**Proof.** Since  $T$  is continuous, there is a  $\delta > 0$  such that for any  $a, b \in E$  satisfying  $\|\frac{a-b}{2}\| \leq \delta$ , we have  $\|T(\frac{a-b}{2})\| \leq \frac{r}{4}$ .

**Step 1.** For any  $a, b \in E$  satisfying  $\|\frac{a-b}{2}\| \leq \delta$ .

Let  $m = T(\frac{a-b}{2})$  and define  $M := \{y \in T(E) : \|y\| = \|2m - y\| \leq 2\|m\| \leq \frac{r}{2}\}$  and  $V : M \rightarrow M$  by  $V(y) = 2m - y$ . Then we get

(i)  $M \neq \emptyset$  since  $m \in M$ .

(ii)  $M$  is a bounded metric space.

(iii)  $V$  is an isometry from  $M$  onto  $M$  since  $V^2 = id_M$  and  $\|V(y_1) - V(y_2)\| = \|2m - y_1 - 2m + y_2\| = \|y_1 - y_2\|, \forall y_1, y_2 \in M$ .

(iv)  $d(Vy, y) = \|Vy - y\| = \|2(m - y)\| \geq K\|m - y\|$  with  $K = \alpha_F(r) > 1$ .

From (i)-(iv) we have shown the conditions of Lemma 1 are satisfied, so  $m = T(\frac{a-b}{2})$  is a fixed point for every surjective isometry of  $M$ .

Since  $2T(\frac{a-b}{2}) \in T(E)$ , we can find an  $x_0 \in E$  such that

$$T(x_0) = 2T\left(\frac{a-b}{2}\right) = 2m. \quad (2)$$

Define  $S : M \rightarrow M$  by  $S(y) = T(x_0 - T^{-1}(y))$ . We shall prove that  $S$  is well-defined and it is an isometry from  $M$  onto  $M$ .

(i)  $S$  is well-defined.

If  $T(x_1) = T(x_2) = y$ , then

$$\|T(x_0 - x_1) - T(x_0 - x_2)\| = \phi\|x_1 - x_2\| = \|Tx_1 - Tx_2\| = 0.$$

(ii)  $S$  is an isometry from  $M$  onto  $M$ .

For any  $y_1, y_2 \in M$  with  $T(x_1) = y_1$  and  $T(x_2) = y_2$ , we have

$$\|Sy_1 - Sy_2\| = \|T(x_0 - x_1) - T(x_0 - x_2)\| = \phi(\|x_1 - x_2\|) = \|Tx_1 - Tx_2\| = \|y_1 - y_2\|.$$

For any  $y \in M \subseteq T(E)$  with  $T(x) = y$ , we have

$$S(y) = T(x_0 - x) \text{ and } \|Tx\| = \|2m - Tx\|. \quad (3)$$

By (2) and (3), we have

$$\begin{aligned} \|Sy\| &= \|T(x_0 - x)\| = \|T(x_0 - x) - T(0)\| = \phi(\|x_0 - x\|) = \|Tx_0 - Tx\| \\ &= \|2m - Tx\| = \|Tx\| = \|T(x) - T(0)\| = \phi(\|x - 0\|) = \phi(\|x_0 - (x_0 - x)\|) \\ &= \|T(x_0) - T(x_0 - x)\| = \|2m - Sy\|. \end{aligned}$$

This implies  $S(M) \subseteq M$  and by the definition of  $S$ , we can easily get  $S^2 = id_M$ , so  $S(M) = (M)$ . Thus  $m$  is a mixed point of  $S$ , i.e.

$$T\left(\frac{a-b}{2}\right) = m = Sm = T\left(x_0 - \frac{a-b}{2}\right). \quad (4)$$

By (2) and (4), we have

$$\begin{aligned} \|T(a-b) - 2T\left(\frac{a-b}{2}\right)\| &= \|T(a-b) - Tx_0\| = \phi(\|a-b - x_0\|) \\ &= \phi\left(\left\|\frac{a-b}{2} - \left(x_0 - \frac{a-b}{2}\right)\right\|\right) \\ &= \left\|T\left(\frac{a-b}{2}\right) - T\left(x_0 - \frac{a-b}{2}\right)\right\| = 0. \end{aligned}$$

This implies

$$T(a - b) = 2T\left(\frac{a - b}{2}\right).$$

For the fixed  $b$ , define  $T_b : E \rightarrow F$  by  $T_b(x) = T(x + b) - T(b)$  and  $T_b$  has the following properties:

(i)  $T_b$  is a continuous map which preserves equality of distance with  $T_b(0) = 0$  and It has the same gauge function  $\phi$  with  $T$ .

This follows from

$$\begin{aligned} \|T_b(x) - T_b(y)\| &= \|T(x + b) - T(b) - T(y + b) + T(b)\| \\ &= \|T(x + b) - T(y + b)\| = \phi(\|x - y\|). \end{aligned}$$

(ii)  $2T_b(x) - T_b(y) \in T_b(E)$ ,  $\forall x, y \in E$

For any  $x, y \in E$ , we have

$$2T_b(x) - T_b(y) = 2T(x + b) - T(y + b) - T(b).$$

Since  $2T(x + b) - T(y + b) \in E$ , there is a  $z_0 \in E$  such that  $T(z_0) = 2T(x + b) - T(y + b)$ . Namely

$$2T_b(x) - T_b(y) = T(z_0) - T(b) = T(z_0 - b + b) - T(b) = T_b(z_0 - b) \in T_b(E).$$

$$(iii) \|T_b\left(\frac{a-b}{2}\right)\| = \phi\left(\left\|\frac{a-b}{2}\right\|\right) = \|T\left(\frac{a-b}{2}\right)\| \leq \frac{r}{4}.$$

This follows from (i).

By (i), (ii) and (iii), we have showed that  $T_b$  has the same conditions with  $T$ . Thus, similarly, we get

$$T_b(a - b) = 2T_b\left(\frac{a - b}{2}\right). \quad (5)$$

By (5), we have

$$\begin{aligned} T\left(\frac{a+b}{2}\right) &= T\left(\frac{a-b}{2} + b\right) - T(b) + T(b) \\ &= T_b\left(\frac{a-b}{2}\right) + T(b) = \frac{T_b(a-b)}{2} + T(b) \\ &= \frac{T(a-b+b) - T(b)}{2} + T(b) = \frac{T(a) + T(b)}{2}. \end{aligned}$$

Thus we have

$$2T\left(\frac{a+b}{2}\right) = T(a) + T(b), \quad \text{for all } a, b \in E \text{ with } \left\| \frac{a-b}{2} \right\| \leq \delta. \quad (6)$$

**Step 2.** Consider the case of  $\left\| \frac{a-b}{2} \right\| > \delta$ . Since  $\| \cdot \|$  is continuous for the multiplication by scalar, there is a positive integer  $N$  such that

$$\left\| \frac{a-b}{2N} \right\| \leq \delta. \quad (7)$$

Next we shall show by induction that

$$2T\left(\frac{a+b}{2}\right) = T\left(\frac{a+b}{2} + \frac{n(a-b)}{2N}\right) + T\left(\frac{a+b}{2} - \frac{n(a-b)}{2N}\right), \quad \forall n \geq 1. \quad (8)$$

Putting  $n = 1$  in (8), by (6) and (7), we find that (8) holds for  $n = 1$ . Let us suppose (8) holds for  $n \leq k$ , then we get

$$2T\left(\frac{a+b}{2}\right) = T\left(\frac{a+b}{2} + \frac{k(a-b)}{2N}\right) + T\left(\frac{a+b}{2} - \frac{k(a-b)}{2N}\right). \quad (9)$$

$$2T\left(\frac{a+b}{2}\right) = T\left(\frac{a+b}{2} + \frac{(k-1)(a-b)}{2N}\right) + T\left(\frac{a+b}{2} - \frac{(k-1)(a-b)}{2N}\right). \quad (10)$$

and by (6) and (7), similarly, we can get

$$2T\left(\frac{a+b}{2} + \frac{k(a-b)}{2N}\right) = T\left(\frac{a+b}{2} + \frac{(k+1)(a-b)}{2N}\right) + T\left(\frac{a+b}{2} + \frac{(k-1)(a-b)}{2N}\right). \quad (11)$$

$$2T\left(\frac{a+b}{2} - \frac{k(a-b)}{2N}\right) = T\left(\frac{a+b}{2} - \frac{(k+1)(a-b)}{2N}\right) + T\left(\frac{a+b}{2} - \frac{(k-1)(a-b)}{2N}\right). \quad (12)$$

Combining (9), (10), (11) and (12), we get

$$2T\left(\frac{a+b}{2}\right) = T\left(\frac{a+b}{2} + \frac{(k+1)(a-b)}{2N}\right) + T\left(\frac{a+b}{2} - \frac{(k+1)(a-b)}{2N}\right).$$

Thus we have proved (8). Putting  $n = N$  in (8), we get

$$T\left(\frac{a+b}{2}\right) = \frac{T(a) + T(b)}{2}.$$

From step 1 and step 2, we get

$$T(a) = T\left(\frac{2a+0}{2}\right) = \frac{T(2a) + T(0)}{2} = \frac{T(2a)}{2} \quad \forall a \in E,$$

and

$$T(a+b) = T\left(\frac{2a+2b}{2}\right) = \frac{T(2a) + T(2b)}{2} = T(a) + T(b), \quad \forall a, b \in E.$$

Thus  $T$  is additive. Since it is continuous and the spaces are real, it is a linear operator and the proof is complete.  $\square$

**Remark 1.** Theorem 2.2 shows that the assumption in [10] which is  $\frac{T(x)+T(y)}{2} \in T(E)$ , for all  $x, y \in E$  can be dropped and the assumptions about the gauge function  $\phi$  for  $T$  can be weakened a lot.

**Remark 2.** If  $T(E)$  is a additive group of  $F$  and keeps the other hypothesis of Theorem 2.1, then  $T$  is linear.

**Corollary 2.3.** Let  $E, F$  be  $p$ -normed spaces ( $0 < p \leq 1$ ), Let  $T : E \rightarrow F$  with  $T(0) = 0$  be a continuous map which preserves equality of distance and

$$2T(x) - T(y) \in T(E), \quad \forall x, y \in E,$$

then

- (i)  $T$  is linear,
- (ii)  $T = \beta V$  where  $\beta$  is a real number and  $V$  is an isometry from  $E$  to  $F$ .

**proof.** Since (i) is a result of Theorem 2.1, we just need to prove (ii). Let  $\phi$  be the gauge function for  $T$ . By the linearity of  $T$ , we get

$$\|Tx - Ty\| = \phi(1)\|x - y\|, \quad \forall x, y \in E. \quad (13)$$

If  $\phi(1) = 0$ , then  $T \equiv 0$ . Let  $\beta = 0$ , then (ii) holds for any isometry  $V$  from  $E$  to  $F$ .

If  $\phi(1) \neq 0$ , then let  $\beta = \pm(\phi(1))^{\frac{1}{p}}$  and  $V = T/\beta$ . By (13) and the linearity of  $T$ , we have

$$\begin{aligned} \|Vx - Vy\| &= \left\| \frac{T}{\beta}(x) - \frac{T}{\beta}(y) \right\| = \frac{1}{\phi(1)} \|Tx - Ty\| \\ &= \frac{1}{\phi(1)} \phi(1) \|x - y\| = \|x - y\|. \quad \forall x, y \in E. \end{aligned}$$

This implies  $V$  is an isometry and the proof is complete.  $\square$

In particular, when  $\phi$  is the identity function, i.e.  $\phi(t) = t$ , for all  $t \geq 0$ . By Theorem 2.2, we get the following result which gives a condition for the linearity of not necessarily surjective isometries between two  $F^*$ -spaces.

**Corollary 2.4.** Let  $E, F$  be  $F^*$ -spaces, and there exists a positive number  $r$  such that  $\alpha_F(r) = \inf\{\frac{\|2x\|}{\|x\|} : x \in F, 0 < \|2x\| \leq r\} > 1$ , Let  $V : E \rightarrow F$  with  $V(0) = 0$  be an isometry, i.e.

$$\|Vx - Vy\| = \|x - y\| \quad \forall x, y \in E.$$

and satisfy

$$2V(x) - V(y) \in V(E) \quad \forall x, y \in E.$$

Then  $V$  is linear.

The following remark was also shown in [10].

**Remark 3.** Let  $E, F$  be normed spaces. Let  $V : E \rightarrow F$  be an isometry with  $V(0) = 0$ . If  $F$  is strictly convex, then

$$2V(x) - V(y) \in V(E), \quad \forall x, y \in E$$

(Thus  $V$  is linear).

**Proof.** For any  $x, y \in E$ , we have

$$\|V(2x - y) - V(x)\| = \|x - y\| = \|Vx - Vy\|, \quad (14)$$

$$\|V(2x - y) - V(y)\| = 2\|x - y\| = 2\|Vx - Vy\|. \quad (15)$$

By (14) and (15), we get

$$\begin{aligned} 2\|V(x) - V(y)\| &= \|V(2x - y) - V(y)\| = \|V(2x - y) - V(x) + V(x) - V(y)\| \\ &\leq \|V(2x - y) - V(x)\| + \|V(x) - V(y)\| = 2\|V(x) - V(y)\|. \end{aligned}$$

Thus

$$\|V(2x - y) - V(x) + V(x) - V(y)\| = \|V(2x - y) - V(x)\| + \|V(x) - V(y)\|.$$

Since  $F$  is strictly convex, there is a  $\lambda > 0$  such that

$$V(2x - y) - V(x) = \lambda\{V(x) - V(y)\}.$$

By (14), we have  $\lambda = 1$ . Namely  $2V(x) - V(y) = V(2x - y) \in V(E)$ .  $\square$

By the remark, we can consider the result of Baker [11] on isometries in strictly convex normed spaces as a consequence of corollary 2.4.

**Remark 4.** Following [13], we give an example to show that Corollary 2.4 is no longer true if  $V$  is an isometry without the hypothesis  $2V(x) - V(y) \in V(E)$ , for all  $x, y \in E$ .

Let  $E = (R^2, \|\cdot\|_p)$ ,  $F = (R^3, \|\cdot\|_p)$  be  $p$ -normed spaces ( $0 < p \leq 1$ ) equipped with  $p$ -normed defined by

$$\|(\xi, \eta)\|_p = (\max(|\xi|, |\eta|))^p, \quad \forall x = (\xi, \eta) \in R^2,$$

$$\|(\xi, \eta, \zeta)\|_p = (\max(|\xi|, |\eta|, |\zeta|))^p, \quad \forall x = (\xi, \eta, \zeta) \in R^3.$$

Let  $V : E \rightarrow F$  be defined by  $V(\xi, \eta) = (\sin \xi, \xi, \eta)$ . It is easy to verify that  $V$  is an isometry which is not linear, and if we let  $\xi_1 = \frac{\pi}{2}$ ,  $\xi_2 = \frac{\pi}{3}$ , we find that  $2V(\xi_1, \eta) - V(\xi_2, \eta) \notin V(E)$ ,  $\forall \eta \in R$ .

## Acknowledgment

The author is grateful to Professor Ding Guanggui for his encouraging advice and many useful suggestions.

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